

BPS center vortices in nonrelativistic $SU(N)$ gauge models with adjoint Higgs fields

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November 26, 2015

Abstract

In this work, we propose a class of $SU(N)$ Yang-Mills models, with adjoint Higgs fields, that accept BPS center vortex equations. The lack of a *local* magnetic flux that could serve as an energy bound is circumvented by including a new term in the energy functional. This term tends to align, in the Lie algebra, the magnetic field and one of the adjoint Higgs fields. Finally, a reduced set of equations for the center vortex profile functions is obtained (for $N = 2, 3$). In particular, $Z(3)$ BPS vortices come in three colours and three anticolours, obtained from an ansatz based on the defining representation and its conjugate.

Keywords:

Pacs: 11.15.-q, 11.10.Lm, 11.15.Kc

1 Introduction

Topological solitons are present in many areas of Physics. Some well-known examples are: kinks in polyacetylene [1], vortices in type II superconductors, skyrmions in magnetic systems [2], and skyrmions to describe baryons in flavour symmetric models [3, 4]. To gain information about these objects, it is important to identify a critical point where a BPS bound is obtained. Namely, a point where the energy can be written as a sum of squares plus the topological charge of the field configuration. As continuous field deformations cannot modify this charge, setting the squares to zero leads to a

set of (BPS) equations whose solutions are absolute minima in the given topological sector. In this process, the equations are reduced to first order, which facilitates analytical and numerical studies of these systems. In addition, BPS multisoliton solutions with a given total charge have the same energy, so the forces between BPS solitons vanish. For these reasons, the critical point provides a nice reference to introduce perturbations and study the soliton dynamics [5].

Topological solitons are also important in effective descriptions of the strong interactions. Abelian Higgs models have been proposed to describe the $q\bar{q}$ potential [6, 7] and the interaction among three quarks [8]-[10]. In refs. [12]-[15], center vortices were accommodated in $SU(N)$ Yang-Mills models with N adjoint Higgs fields; these objects can describe the N-ality properties of the confining string [11]. Recently, we proposed a class of flavour symmetric models supporting not only the confining string between a $q\bar{q}$ colourless pair of external quarks, but also other possible excited states [16]. Among them, $qq\bar{q}'$ hybrid mesons [17]-[19], formed for example by a red/anti-green pair of quarks bound by an anti-red/green valence gluon. While the normal string is a center vortex of the effective model, the excited string is formed by a pair of center vortices interpolated by a monopole, which is identified with a confined valence gluon.

The topology and classification of center vortices have been analyzed in ref. [11], when a general compact gauge group G is broken down to its center. The roots of the Lie algebra and the weights of their representations play an important role, as occurs when characterizing non Abelian monopoles [20]. BPS equations for non Abelian vortices have been obtained in refs. [21]-[23], for a review, see refs. [24, 25].

In ref. [16], we proposed a Lorentz invariant flavour symmetric model that is expected to contain center vortices, as it possesses the proper SSB pattern and topology. The problem is that the field equations are mathematically difficult. They can only be solved by following numerical methods. With the aim of exploring the usual tools to understand topological objects, in this work we shall look for models accepting BPS center vortices, governed by *first order* field equations. As an intermediate step, we shall simplify the content of the flavour symmetric model, which is based on $N^2 - 1$ adjoint Higgs fields that form a local Lie basis at the nontrivial vacua of the Higgs potential. Observing that the essential features of the Lie algebra can be captured by a reduced set of fields and conditions, labelled by the simple roots, a Lorentz invariant model that for $N \geq 3$ has a simplified field content will be obtained. The Higgs potential will be such that its minimization returns a set of conditions that essentially define a Chevalley basis. This model has the same SSB pattern and topology than the former. Next, we shall

make an extension to obtain a model that accepts BPS center vortices. In this process, we can anticipate some peculiarities. Generally, BPS equations are derived by working on the energy functional to obtain a bound (for an alternative approach, see ref. [26]). For $U(1)$ vortices, the bound is given by the magnetic flux. This is a topological term that can be written locally, by means of a flux density, so it can indeed arise by working on the energy, which is a local functional. On the other hand, for center vortices, the flux concept is replaced by the asymptotic behavior of the gauge invariant Wilson loop, a nonlocal object that may not appear in the energy calculation. Then, the search for BPS equations in $SU(N) \rightarrow Z(N)$ SSB models led us to consider the introduction of a nonrelativistic interaction term that tends to align, in the Lie algebra, the magnetic field along one of the adjoint Higgs fields. This in turn implied a different type of bound. After completing the squares, the energy is always greater than or equal to zero. Thus, BPS center vortices are nonrelativistic objects characterized by an exact compensation between the positive definite part of the energy functional (kinetic energy plus Higgs potential) and the Lie algebra alignment contribution.

In this regard, two comments are in order: i) The reason for considering the intermediate step is that, for $N \geq 3$, the direct inclusion of the alignment term in the flavour symmetric model would lead, after completing the squares, to too many (possibly incompatible) conditions to saturate the bound. ii) Because of Lorentz symmetry breaking, and rotational symmetry breaking in $3 + 1$ dimensions, the BPS models are not directly physically relevant. However, the presence of a BPS point in the extended parameter space could serve as a check for the numerical analysis, when moving away from the physically relevant non BPS Lorentz invariant confining models.

The general BPS solution will be written in terms of a set of profile functions and a mapping $R(S)$ in the adjoint representation of $SU(N)$. The mapping $S \in SU(N)$ contains information about the asymptotic Wilson loop and the possible defects at the vortex guiding centers, which determine the profile behaviours. Because of the model's topology, a given phase $S_0(\varphi)$, defined close to and around a vortex guiding center, can be extended to different asymptotic phases $S_a(\varphi)$, where $S_a(\varphi + 2\pi) = e^{i2\pi z_a/N} S_a(\varphi)$, $z_a \in \mathbb{Z}$. The $Z(N)$ charge is due to the fact that the different extensions are related by $(z_{a'} - z_a)/N \in \mathbb{Z}$. For the same reason, a given S_a can be matched with different phases S_0 , with their respective pointlike defects. When leaving the critical point, by lowering the alignment interaction term, some of these extensions will become unstable. For example, for vanishing $Z(N)$ -charge the defect can be avoided, and the lowest energy solution will simply correspond to a trivial regular gauge transformation of the SSB vacua. For $Z(N)$ charge ± 1 , we shall discuss the BPS solutions that are expected to be related to the

stable $Z(2)$ and $Z(3)$ noncritical center vortices.

The article is organized as follows. In section 2, we construct the simplified $SU(N)$ model and discuss the possible vacua. In section 3, we obtain the bounds and the set of BPS equations (for $N = 2, 3$). Some properties of the field parametrization are discussed in section 4. Section 5 is devoted to obtaining information about the BPS solutions and discussing the BPS center vortex. Finally, in section 6, we present our conclusions.

2 Models with $SU(N) \rightarrow Z(N)$ SSB

In order to support center vortices, we are interested in driving a phase where the gauge symmetry is spontaneously broken down to $Z(N)$. For example, in ref. [16], we introduced a model displaying a flavour symmetry. That is, we considered the energy functional¹,

$$E = \int d^3x \left(\frac{1}{2} \langle B_i \rangle^2 + \frac{1}{2} \langle D_i \psi_A \rangle^2 + V_{\text{Higgs}}(\psi_A) \right), \quad (1)$$

where B_i is the chromomagnetic field, $D_i = \partial_i - ig[A_i, \]$, and the potential for the hermitian adjoint Higgs fields ψ_A , $A = 1, \dots, d = N^2 - 1$, is given by,

$$V_{\text{Higgs}} = c + \frac{\mu^2}{2} \langle \psi_A, \psi_A \rangle + \frac{\kappa}{3} f_{ABC} \langle (-i)[\psi_A, \psi_B], \psi_C \rangle + \frac{\lambda}{4} \langle [\psi_A, \psi_B], [\psi_A, \psi_B] \rangle, \quad (2)$$

where f_{ABC} are structure constants of the $\mathfrak{su}(N)$ Lie algebra. At $\mu^2 = \frac{2}{9} \frac{\kappa^2}{\lambda}$, $\kappa < 0$, we can write,

$$V_{\text{Higgs}} = \frac{\lambda}{4} \langle \Psi_{AB} \rangle^2, \quad \Psi_{AB} = f_{ABC} v_c \psi_C + i[\psi_A, \psi_B], \quad (3)$$

$$v_c = -\frac{\kappa}{2\lambda} \pm \sqrt{\left(\frac{\kappa}{2\lambda}\right)^2 - \frac{\mu^2}{\lambda}} = -\frac{2\kappa}{3\lambda}, \quad (4)$$

after adjusting c , so that the potential energy for vacuum configurations vanishes. The space of vacua \mathcal{M} is obtained from the conditions $\Psi_{AB} = 0$, i.e.,

$$[\psi_A, \psi_B] = i f_{ABC} v_c \psi_C. \quad (5)$$

This encompasses the trivial symmetric point $\psi_A = 0$, separated by a potential barrier from the nontrivial points. Of course, starting from a nontrivial

¹We are using the inner product $\langle X, Y \rangle = \text{Tr} (Ad(X)^\dagger Ad(Y))$, where $Ad(\cdot)$ is a linear map into the adjoint representation.

point $\psi_A \in \mathcal{M}$, we can generate a continuum $S\psi_AS^{-1}$, $S \in SU(N)$, that is also in \mathcal{M} . In addition, the only transformations that leave these points invariant are $S \in Z(N)$, so they correspond to $SU(N) \rightarrow Z(N)$ SSB vacua. For $N \geq 3$, the SSB points can be divided into a pair of distinct sets, separated by a potential barrier, corresponding to the defining representation and its conjugate,

$$\psi_A = ST_AS^{-1} \quad , \quad \psi_A = S(-T_A)^*S^{-1} . \quad (6)$$

For $N = 2$, this pair collapses into a single component, as a matrix $S_c \in SU(2)$ exists such that $(-T_A)^* = S_c T_A S_c^{-1}$, $A = 1, 2, 3$.

Although the model in eq. (1) contains center vortices, we did not succeed in taking it as a starting point to obtain BPS equations (for $N \geq 3$). The presence of too many fields ultimately leads to incompatible conditions to saturate the bound. For this reason, in the next section, we shall look for a simplified model. Instead of the previous $N^2 - 1$ hermitian fields, we introduce $N - 1$ hermitian and $N - 1$ complex adjoint Higgs fields. For $N \geq 3$, this will result in a simpler set of fields and conditions to define the SSB vacua. This, together with the “alignment” term introduced in section 3, will finally lead to a set of compatible BPS equations.

2.1 Simplified model

Let us consider hermitian variables, ψ_q , $q = 1, \dots, r = N - 1$, and complex variables ζ_α , labelled by the positive simple roots $\vec{\alpha}_q$ ($\vec{\alpha}_1 < \vec{\alpha}_2 < \dots < \vec{\alpha}_r$). The conditions,

$$[\psi_q, \psi_p] = 0 \quad , \quad v_c \vec{\alpha}|_q \zeta_\alpha - [\psi_q, \zeta_\alpha] = 0 , \quad (7)$$

contain most of the relevant structure of the Lie algebra. For nontrivial fields ζ_α , we can imply,

- i) The fields ψ_q are nontrivial, as their sizes are fixed by the eigenvalues $v_c \vec{\alpha}|_q$. They are also linearly independent: if there is a combination $\gamma^q \psi_q = 0$, then using eq. (7) we get $\vec{\gamma} \cdot \vec{\alpha} = 0$, for every simple root, so that $\vec{\gamma} = 0$.
- ii) $[\zeta_\alpha, \zeta_\alpha^\dagger]$ is in the Cartan subalgebra generated by the fields ψ_q .
- iii) As the positive (negative) roots can be written as a linear combination of the simple roots, with nonnegative (nonpositive) integer coefficients, any root vector is proportional to an appropriate chain of operations of the form $[\zeta_\alpha, \zeta_{\alpha'}] ([\zeta_\alpha^\dagger, \zeta_{\alpha'}^\dagger])$.

- iv) As the difference of a pair of positive simple roots cannot be a root, we have $[\zeta_\alpha, \zeta_{\alpha'}^\dagger] = 0$.

However, considering a potential whose minimization only leads to the conditions in eq. (7) would not be the desired one. Given a nontrivial solution (ψ_q, ζ_α) , the replacement $\zeta_\alpha \rightarrow t \zeta_\alpha$, $t \in \mathbb{R}$, would also lead to a solution. Then, the interesting SSB initial point could be continuously moved to $(\psi_q, 0)$, and then ψ_q could be continuously moved to 0, always staying in the space of vacua \mathcal{M} . That is, there would be no potential barrier between the interesting configurations and the trivial one. This will be corrected by including a term in the potential to avoid, after minimization, the possibility of moving the fields ζ to zero, when we start with a SSB point. For this purpose, let us consider the additional condition,

$$\sum_q s_q \left(v_c \vec{\alpha}_q \cdot \vec{\psi} - [\zeta_{\alpha_q}, \zeta_{\alpha_q}^\dagger] \right) = 0 \quad , \quad \vec{\alpha} \cdot \vec{\psi} = \vec{\alpha}|_{q'} \psi_{q'} \quad , \quad (8)$$

where s_q takes values $+1$ or -1 . Now, we consider a solution to eq. (7), and recall that given linearly independent fields ψ_q it is always possible to introduce unique elements \mathcal{H}_α such that,

$$\langle \mathcal{H}_\alpha, \psi_q \rangle = v_c \vec{\alpha}|_q \quad . \quad (9)$$

As is well known [27]-[29], these variables satisfy,

$$[\zeta_\alpha, \zeta_\alpha^\dagger] = \langle \zeta_\alpha, \zeta_\alpha \rangle \mathcal{H}_\alpha \quad . \quad (10)$$

Using this information in eq. (8) and projecting with \mathcal{H}_{α_p} , we get,

$$\begin{aligned} \sum_q s_q \left(v_c \vec{\alpha}_q|_{q'} \langle \mathcal{H}_{\alpha_p}, \psi_{q'} \rangle - \langle \zeta_{\alpha_q}, \zeta_{\alpha_q} \rangle \langle \mathcal{H}_{\alpha_p}, \mathcal{H}_{\alpha_q} \rangle \right) = \\ = \sum_q s_q \left(v_c^2 \vec{\alpha}_q \cdot \vec{\alpha}_p - \langle \zeta_{\alpha_q}, \zeta_{\alpha_q} \rangle \langle \mathcal{H}_{\alpha_p}, \mathcal{H}_{\alpha_q} \rangle \right) = 0 \quad . \end{aligned} \quad (11)$$

From the Lie algebra internal product and the mapping $\vec{\alpha} \rightarrow \mathcal{H}_\alpha$, an internal product on the root space can be defined [27]-[29],

$$\langle \vec{\alpha}, \vec{\alpha}' \rangle \equiv \langle \mathcal{H}_\alpha, \mathcal{H}_{\alpha'} \rangle \quad . \quad (12)$$

These quantities are strongly constrained. In particular,

$$2 \langle \vec{\alpha}, \vec{\alpha}' \rangle / \langle \vec{\alpha}', \vec{\alpha}' \rangle \in \mathbb{Z} \quad , \quad (13)$$

are the so-called Cartan integers, which determine the geometry of the root lattice. They do not depend on the Cartan basis, coinciding with,

$$2(\vec{\alpha} \cdot \vec{\alpha}')/(\vec{\alpha}' \cdot \vec{\alpha}') , \quad (14)$$

which corresponds to (13), when computed with an orthogonal basis ψ_q , $\langle \psi_q, \psi_p \rangle = v_c^2 \delta_{qp}$. Note that in this case, $\mathcal{H}_\alpha = (1/v_c) \vec{\alpha}|_q \psi_q$. In addition, for $\mathfrak{su}(N)$, the lengths of the roots are equal, $\langle \vec{\alpha}, \vec{\alpha} \rangle = c$, $\vec{\alpha} \cdot \vec{\alpha} = 1/N$. Then, using this information, eq. (11) implies,

$$\sum_q s_q (v_c^2 \vec{\alpha}_q \cdot \vec{\alpha}_p - Nc \langle \zeta_{\alpha_q}, \zeta_{\alpha_q} \rangle \vec{\alpha}_q \cdot \vec{\alpha}_p) = 0 . \quad (15)$$

This is valid for any basis element $\vec{\alpha}_p$, that is,

$$\sum_q s_q (v_c^2 - Nc \langle \zeta_{\alpha_q}, \zeta_{\alpha_q} \rangle) \vec{\alpha}_q = 0 , \quad (16)$$

and as the simple roots $\vec{\alpha}_q$ are linearly independent, we get,

$$\langle \zeta_{\alpha_q}, \zeta_{\alpha_q} \rangle = \frac{v_c^2}{Nc} . \quad (17)$$

This means that if we define the space of vacua \mathcal{M} by means of the conditions (7) and (8), a nontrivial SSB point $(\psi_q, \zeta_\alpha) \in \mathcal{M}$ cannot be continuously moved to the trivial solution $(0, 0)$, always staying in \mathcal{M} . In effect, starting with nontrivial fields ζ_α implies linearly independent fields ψ_q , and this in turn leads to nontrivial \mathcal{H}_α that protect the size of ζ_α through eq. (17). In other words, a potential whose minimization gives the conditions (7) and (8) has a barrier between the SSB vacua and the trivial one.

It is convenient to introduce a model without referring to a particular convention for the simple roots. The field $\vec{\psi}$ with $r = N - 1$ components, such that $\vec{\psi}|_q = \psi_q$, can be expanded either in terms of the simple roots $\vec{\alpha}_q$ or the $\vec{\lambda}^q$ basis satisfying,

$$\vec{\alpha}_q \cdot \vec{\lambda}^p = \delta_q^p \quad , \quad \vec{\alpha}_q|_p \vec{\lambda}^q|^{p'} = \delta_p^{p'} , \quad (18)$$

$$\vec{\psi} = \vec{\alpha}_q \phi^q \quad \text{with,} \quad \phi^q = \vec{\lambda}^q \cdot \vec{\psi} , \quad (19)$$

$$\vec{\psi} = \vec{\lambda}^q \phi_q \quad \text{with,} \quad \phi_q = \vec{\alpha}_q \cdot \vec{\psi} , \quad (20)$$

$$\psi_p = \vec{\alpha}_q|_p \phi^q = \vec{\lambda}^q|_p \phi_q . \quad (21)$$

For $\mathfrak{su}(N)$, the $\vec{\lambda}^q$ basis is given by,

$$\vec{\lambda}^q = 2N \vec{\Lambda}^q , \quad (22)$$

where $\vec{\Lambda}^q$ are the fundamental weights (see appendix A).

The relation between the different components is,

$$\phi_q = A_{qp} \phi^p \quad , \quad \phi^q = A^{qp} \phi_p \quad , \quad (23)$$

$$A_{qp} = \vec{\alpha}_q \cdot \vec{\alpha}_p \quad , \quad A^{qp} = \vec{\lambda}^q \cdot \vec{\lambda}^p \quad . \quad (24)$$

For $\mathfrak{su}(N)$, the quantities $C_{qp} = 2N A_{qp}$ are the elements of the Cartan matrix, which define the natural product in the root space.

With these definitions, together with $\zeta_q = \zeta_{\alpha_q}$, the conditions in eqs. (7) and (8) become,

$$[\phi_q, \phi_p] = 0 \quad , \quad v_c \delta_{(p)}^q \zeta_{(p)} - [\phi^q, \zeta_p] = 0 \quad , \quad (25)$$

$$\sum_q s_q \left(v_c \phi_q - [\zeta_{(q)}, \zeta_{(q)}^\dagger] \right) = 0 \quad , \quad (26)$$

which can be obtained by minimizing the Higgs potential,

$$V_{\text{Higgs}} = \frac{1}{2} \langle \Phi, \Phi \rangle + \langle Z_q, Z_q \rangle + \mathcal{R} \quad , \quad (27)$$

where,

$$\Phi = \sum_q s_q \Phi_q \quad , \quad \Phi_q = \sqrt{\gamma} \left(v_c \phi_q - [\zeta_{(q)}, \zeta_{(q)}^\dagger] \right) \quad , \quad (28)$$

$$Z_q = \sqrt{\gamma_z} \left(v_c \zeta_q - [\phi^{(q)}, \zeta_{(q)}] \right) \quad , \quad (29)$$

$$\mathcal{R} = \gamma_r \sum_{q \neq p} \left(\frac{1}{2} \langle [\phi_q, \phi_p] \rangle^2 + \langle [\phi^q, \zeta_p] \rangle^2 \right) \quad . \quad (30)$$

Now, noting that,

$$\langle D_i \psi_q, D_i \psi_q \rangle = A^{qp} \langle D_i \phi_q, D_i \phi_p \rangle \quad , \quad (31)$$

we initially propose the model,

$$E = \int d^3x \left(\frac{1}{2} \langle B_i \rangle^2 + \frac{1}{2} \langle D_i \phi_q, D_i \phi^q \rangle + \langle D_i \zeta_q, D_i \zeta_q \rangle + V_{\text{Higgs}} \right) \quad . \quad (32)$$

Here, the space of vacua \mathcal{M} is given by the trivial point $\phi_q = 0, \zeta_q = 0$, separated by a potential barrier from the SSB points. For $N \geq 3$, the latter can be separated into the sets,

$$\phi_q = S(\vec{\alpha}_q \cdot \vec{H}) S^{-1} \quad , \quad \zeta_q = S E_{\alpha_q} S^{-1} \quad (33)$$

$$\phi_q = S(-\vec{\alpha}_q \cdot \vec{H}) S^{-1} \quad , \quad \zeta_q = S(-E_{\alpha_q}^T) S^{-1} \quad , \quad (34)$$

where E_{α}^T is the transpose of E_{α} , and H_q, E_{α} are elements of a Cartan basis (see appendix A). For $N = 2$, the sets in (33) and (34) are equal.

3 Nonrelativistic models with BPS center vortex equations

As is well known, center vortices are characterized by a center element

$$\mathfrak{z} = e^{i2\pi z/N} I \in SU(N) \quad (35)$$

such that, for a path linking the vortex and contained in the asymptotic region, the Wilson loop gives,

$$W[A] = \mathfrak{z} . \quad (36)$$

The center vortex has a $Z(N)$ charge given by z , defined modulo N . In particular, this is the case when in an asymptotic region $r > r_m$ the gauge field is given by,

$$A_i = \frac{1}{g} \partial_i \varphi \vec{\beta} \cdot \vec{H} \quad , \quad e^{i2\pi \vec{\beta} \cdot \vec{H}} = \mathfrak{z} , \quad (37)$$

where r and φ are polar coordinates with respect to the vortex axis. The possible magnetic weights $\vec{\beta}$ satisfy

$$\vec{\beta} \cdot \vec{\alpha} \in Z , \quad (38)$$

for every root $\vec{\alpha}$. The solutions to eq. (38) are [20], [14, 11],

$$\vec{\beta} = 2N\vec{w} , \quad (39)$$

where \vec{w} are the weights of the different representations. The minimum charge center vortices ($z = \pm 1$) can be labelled by the weights of the defining representation and its conjugate [11].

In the asymptotic region, if a Higgs field takes the value X_0 , $\langle X_0, X_0 \rangle = v_c^2$ at $\varphi = 0$ then, on the circle at infinity, the non Abelian phase will accompany the A_i behaviour in eq. (37) as follows²,

$$X = SX_0S^{-1} \quad , \quad S = e^{i\varphi \vec{\beta} \cdot \vec{T}} . \quad (40)$$

Now, we would like to propose models accepting BPS center vortex equations for $SU(2)$ and $SU(3)$. To simplify the discussion, let us consider planar systems, replacing $d^3x \rightarrow d^2x = dx_1 dx_2$, $B_3 \rightarrow B$, and taking $B_1 = B_2 = 0$. Initially, we note that the type of models we have discussed so far cannot

²we use X to denote any of the Higgs fields ϕ_q, ζ_q .

accept a BPS bound. Indeed, this would be the case in any model whose energy functional only vanishes for vacuum configurations, while on the space of field configurations $\{A, X\}_{\mathfrak{z}}$, with a given nontrivial asymptotic behaviour labelled by \mathfrak{z} , it is strictly positive. In this case, to obtain BPS center vortex equations, the energy functional should be bounded by a nonzero term verifying: i) gauge invariance, ii) it assumes a fixed value on the space $\{A, X\}_{\mathfrak{z}}$, that only depends on \mathfrak{z} (topological), iii) as the bound would be derived by working on the energy density, it should have the form $\int d^2x \rho$ (locality). While the Wilson loop verifies i) and ii), it is a nonlocal object that cannot arise in the calculation. On the other hand, while

$$\int d^2x \langle \eta, B \rangle \quad , \quad B = \partial_1 A_2 - \partial_2 A_1 - ig[A_1, A_2] \quad , \quad (41)$$

with η an adjoint field, satisfies i) and iii), it does not satisfy ii). This would be a boundary term for homogeneous η and those Abelian-like fields in $\{A, X\}_{\mathfrak{z}}$ such that $B = \partial_1 A_2 - \partial_2 A_1$ on the *whole* plane. Then, the search for BPS center vortices should consider a modified class of models where configurations in $\{A, X\}_{\mathfrak{z}}$ do not necessarily have strictly positive energy.

For example, we will see that the model (32) could be reorganized as a sum of squares plus a term of the form (41). As this term does not satisfy ii), setting the squares to zero will not produce, in a given sector $\{A, X\}_{\mathfrak{z}}$, solutions to the field equations associated with (32). Then, it is natural to try a modified model,

$$E = \int d^2x \left(\frac{1}{2} \langle B \rangle^2 + \frac{1}{2} \langle D_i \phi_q, D_i \phi^q \rangle + \langle D_i \zeta_q, D_i \zeta_q \rangle + V_{\text{Higgs}} - \langle \eta, B \rangle \right) . \quad (42)$$

Here, we have included the gauge invariant $\langle \eta, B \rangle$ -interaction, that tends to align B along η in the Lie algebra. The field η will be an appropriate combination of the adjoint Higgs fields, to be determined in order for the model to accept BPS center vortex equations. At the critical point, we shall see that in spite of the last term in eq. (42), this energy functional satisfies $E \geq 0$. For BPS solutions, the contribution originated from the positive definite terms will be exactly compensated by the energy lowering due to the Lie algebra alignment between magnetic and Higgs fields. Thus, the topologically nontrivial BPS center vortices will have vanishing energy. We note that with this term the planar model becomes nonrelativistic although it continues to be isotropic in $2 + 1$ dimensions³.

Let us derive the fundamental property to discuss BPS bounds. Using

³In $3 + 1$ dimensions, this type of model would also break rotation symmetry.

the ciclicity of the internal product,

$$\langle X, [Y, Z] \rangle = \langle [X, Z^\dagger], Y \rangle , \quad (43)$$

as A_i is hermitian, we have,

$$\begin{aligned} \langle D_i X, Y \rangle &= \langle \partial_i X - ig[A_i, X], Y \rangle \\ &= \partial_i \langle X, Y \rangle - \langle X, \partial_i Y \rangle + ig \langle [A_i, X], Y \rangle \\ &= \partial_i \langle X, Y \rangle - \langle X, D_i Y \rangle , \end{aligned} \quad (44)$$

$$\langle D_i X, Y \rangle + \langle X, D_i Y \rangle = \partial_i \langle X, Y \rangle . \quad (45)$$

Now, defining,

$$D = D_1 + iD_2 , \quad (46)$$

we note that,

$$\begin{aligned} \langle DX, DX \rangle &= \langle D_1 X + iD_2 X, D_1 X + iD_2 X \rangle \\ &= \langle D_1 X, D_1 X \rangle + \langle D_2 X, D_2 X \rangle - i \langle D_2 X, D_1 X \rangle + i \langle D_1 X, D_2 X \rangle . \end{aligned} \quad (47)$$

In addition, as B is hermitian,

$$\langle X, [B, X] \rangle = \langle [X, X^\dagger], B \rangle = \langle B, [X, X^\dagger] \rangle . \quad (48)$$

This together with eq. (44) and,

$$[D_\mu, D_\nu] X = -ig[F_{\mu\nu}, X] , \quad (49)$$

which is obtained from the Jacobi identity, we get,

$$\begin{aligned} \langle D_2 X, D_1 X \rangle - \langle D_1 X, D_2 X \rangle &= \\ &= -\langle X, D_2 D_1 X \rangle + \langle X, D_1 D_2 X \rangle + \partial_2 \langle X, D_1 X \rangle - \partial_1 \langle X, D_2 X \rangle \\ &= -ig \langle X, [F_{12}, X] \rangle + \partial_2 \langle X, D_1 X \rangle - \partial_1 \langle X, D_2 X \rangle \\ &= -ig \langle B, [X, X^\dagger] \rangle + \partial_2 \langle X, D_1 X \rangle - \partial_1 \langle X, D_2 X \rangle . \end{aligned} \quad (50)$$

Therefore,

$$\begin{aligned} \langle D_i X, D_i X \rangle &= \\ &= \langle DX, DX \rangle + g \langle B, [X, X^\dagger] \rangle + \partial_2 \langle X, iD_1 X \rangle - \partial_1 \langle X, iD_2 X \rangle , \end{aligned} \quad (51)$$

and similarly,

$$\begin{aligned} \langle D_i X, D_i X \rangle &= \\ &= \langle \bar{D} X, \bar{D} X \rangle - g \langle B, [X, X^\dagger] \rangle - \partial_2 \langle X, iD_1 X \rangle + \partial_1 \langle X, iD_2 X \rangle , \end{aligned} \quad (52)$$

$$\bar{D} = D_1 - iD_2 . \quad (53)$$

3.1 $SU(2)$ model

For $SU(2)$, there is simply a one component positive root, $\alpha_1 = \frac{1}{\sqrt{2}}$, and $A_{11} = \frac{1}{2}$, $A^{11} = 2$. Naming $\phi_1 = \phi$, $\zeta_1 = \zeta$, the model in eq. (42) is,

$$E = \int d^2x \left(\frac{1}{2} \langle B \rangle^2 + \langle D_i \phi \rangle^2 + \langle D_i \zeta \rangle^2 + V_{\text{Higgs}} - \langle \eta, B \rangle \right). \quad (54)$$

The Higgs potential can be written as,

$$V_{\text{Higgs}} = \frac{1}{2} \langle \Phi \rangle^2 + \langle Z \rangle^2, \quad (55)$$

$$\Phi = \sqrt{\gamma} (v_c \phi - [\zeta, \zeta^\dagger]) \quad , \quad Z = \sqrt{\gamma_z} (v_c \zeta - 2[\phi, \zeta]) \quad (56)$$

Now, using

$$\langle B \rangle^2 + \langle \Phi \rangle^2 = \langle \Phi - B \rangle^2 + 2\langle \Phi, B \rangle. \quad (57)$$

and the property (51), for $X = \zeta$, namely,

$$\langle D_i \zeta \rangle^2 = \langle D\zeta \rangle^2 + g\langle [\zeta, \zeta^\dagger], B \rangle + \partial_2 \langle \zeta, iD_1 \zeta \rangle - \partial_1 \langle \zeta, iD_2 \zeta \rangle, \quad (58)$$

we obtain,

$$E = \int d^2x \left(\langle D_i \phi \rangle^2 + \langle D\zeta \rangle^2 + \langle Z \rangle^2 + \frac{1}{2} \langle \Phi - B \rangle^2 + \langle \Phi + g[\zeta, \zeta^\dagger] - \eta, B \rangle \right). \quad (59)$$

Here, we have used a boundary condition at $(x^1, x^2) \rightarrow \infty$,

$$D_i \zeta \rightarrow 0 \quad \text{for} \quad (x^1, x^2) \rightarrow \infty. \quad (60)$$

Then, at $\gamma = g^2$ and taking the Lie algebra element,

$$\eta = gv_c \phi, \quad (61)$$

we get,

$$E = \int d^2x \left(\langle D_i \phi \rangle^2 + \langle D\zeta \rangle^2 + \langle Z \rangle^2 + \frac{1}{2} \langle \Phi - B \rangle^2 \right). \quad (62)$$

The bound is saturated when,

$$D_i \phi = 0, \quad (63)$$

$$v_c \zeta - 2 [\phi, \zeta] = 0 , \quad (64)$$

$$D\zeta = 0 , \quad (65)$$

$$B = g (v_c \phi - [\zeta, \zeta^\dagger]) . \quad (66)$$

At the critical point, and taking $\gamma_z = g^2/2$, we can write,

$$V_{\text{Higgs}} = \frac{g^2}{2} [\langle v_c \phi - [\zeta, \zeta^\dagger] \rangle^2 + \langle v_c \zeta - 2 [\phi, \zeta] \rangle^2] . \quad (67)$$

Using $\phi = \alpha_1 \psi_1 = \frac{1}{\sqrt{2}} \psi_1$, and defining,

$$\psi_2 = \frac{\zeta + \zeta^\dagger}{\sqrt{2}} , \quad \psi_3 = \frac{\zeta - \zeta^\dagger}{\sqrt{2}i} , \quad \sigma = \frac{v_c}{\sqrt{2}} , \quad (68)$$

the model accepting BPS solutions is given by,

$$E = \int d^2x \left(\frac{1}{2} \langle B \rangle^2 + \frac{1}{2} \langle D_i \psi_A \rangle^2 + V_{\text{Higgs}} - g v_c \langle \phi, B \rangle \right) . \quad (69)$$

$$V_{\text{Higgs}} = \frac{g^2}{2} [\langle \sigma \psi_1 + i[\psi_2, \psi_3] \rangle^2 + \langle \sigma \psi_2 + i[\psi_3, \psi_1] \rangle^2 + \langle \sigma \psi_3 + i[\psi_1, \psi_2] \rangle^2] , \quad (70)$$

which is a modified version of the flavour symmetric model in eqs. (1), (2).

3.2 $SU(3)$ model

The Higgs potential is,

$$V_{\text{Higgs}} = \frac{1}{2} \langle \Phi, \Phi \rangle + \langle Z_q, Z_q \rangle + \mathcal{R} , \quad (71)$$

$$\Phi = s_1 \Phi_1 + s_2 \Phi_2 , \quad \Phi_q = \sqrt{\gamma} (v_c \phi_q - [\zeta_{(q)}, \zeta_{(q)}^\dagger]) , \quad (72)$$

$$\mathcal{R} = \gamma_r (\langle [\phi_1, \phi_2] \rangle^2 + \langle [\phi^1, \zeta_2] \rangle^2 + \langle [\phi^2, \zeta_1] \rangle^2) . \quad (73)$$

To obtain a set of BPS equations, we initially diagonalize the ϕ_q -kinetic term in eq. (42). Note that any quantity of the form $\langle X_q, X^q \rangle$ can be written as,

$$\langle X_q, X^q \rangle = \frac{1}{2} \langle X_2 + X_1, X^2 + X^1 \rangle + \frac{1}{2} \langle X_2 - X_1, X^2 - X^1 \rangle . \quad (74)$$

On the other hand, the Cartan matrix for $SU(3)$ is,

$$\mathbb{C} = 6\mathbb{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad (75)$$

$\mathbb{C}|_{qp} = C_{qp}$, $\mathbb{A}|_{qp} = A_{qp}$. Therefore,

$$X_2 + X_1 = (A_{22} + A_{12})X^2 + (A_{21} + A_{11})X^1 = \frac{1}{6}(X^2 + X^1), \quad (76)$$

$$X_2 - X_1 = (A_{22} - A_{12})X^2 + (A_{21} - A_{11})X^1 = \frac{1}{2}(X^2 - X^1). \quad (77)$$

That is,

$$\langle X_q, X^q \rangle = 3\langle X_+, X_+ \rangle + \langle X_-, X_- \rangle, \quad X_+ = X_2 + X_1, \quad X_- = X_2 - X_1, \quad (78)$$

and the energy functional in eq. (42) results,

$$\begin{aligned} E = & \int d^2x \left(\frac{1}{2}\langle B \rangle^2 + \frac{3}{2}\langle D_i\phi_+ \rangle^2 + \frac{1}{2}\langle D_i\phi_- \rangle^2 + \langle D_i\zeta_q, D_i\zeta_q \rangle \right) \\ & + \int d^2x \left(\frac{1}{2}\langle \Phi \rangle^2 + \langle Z_q, Z_q \rangle + \mathcal{R} - \langle \eta, B \rangle \right). \end{aligned} \quad (79)$$

Next, similarly to the $SU(2)$ case, using,

$$\langle B \rangle^2 + \langle \Phi \rangle^2 = \langle \Phi - B \rangle^2 + 2\langle \Phi, B \rangle, \quad (80)$$

and properties (51), (52) for $X = \zeta_2, \zeta_1$, respectively,

$$\langle D_i\zeta_2, D_i\zeta_2 \rangle = \langle D\zeta_2, D\zeta_2 \rangle + g\langle [\zeta_2, \zeta_2^\dagger], B \rangle + \partial_3\langle \zeta_2, iD_2\zeta_2 \rangle - \partial_2\langle \zeta_2, iD_3\zeta_2 \rangle,$$

$$\langle D_i\zeta_1, D_i\zeta_1 \rangle = \langle \bar{D}\zeta_1, \bar{D}\zeta_1 \rangle - g\langle [\zeta_1, \zeta_1^\dagger], B \rangle - \partial_3\langle \zeta_1, iD_2\zeta_1 \rangle + \partial_2\langle \zeta_1, iD_3\zeta_1 \rangle,$$

we obtain,

$$\begin{aligned} E = & \int d^2x \left(\frac{3}{2}\langle D_i\phi_+ \rangle^2 + \frac{1}{2}\langle D_i\phi_- \rangle^2 + \langle \bar{D}\zeta_1 \rangle^2 + \langle D\zeta_2 \rangle^2 + \langle Z_q, Z_q \rangle \right) \\ & + \int d^2x \left(\frac{1}{2}\langle \Phi - B \rangle^2 + \langle \Phi + g[\zeta_2, \zeta_2^\dagger] - g[\zeta_1, \zeta_1^\dagger] - \eta, B \rangle \right) \\ & + \int d^2x \gamma_r (\langle [\phi_1, \phi_2] \rangle^2 + \langle [\phi^1, \zeta_2] \rangle^2 + \langle [\phi^2, \zeta_1] \rangle^2), \end{aligned} \quad (81)$$

where we have used that the system is in a local vacuum at $(x^1, x^2) \rightarrow \infty$.

Then, taking $s_1 = -1$, $s_2 = 1$, $\gamma = g^2$, which gives,

$$\Phi = gv_c \phi_- - g \left([\zeta_2, \zeta_2^\dagger] - [\zeta_1, \zeta_1^\dagger] \right) \quad , \quad \phi_- = \phi_2 - \phi_1 \quad , \quad (82)$$

and the Lie algebra element,

$$\eta = gv_c \phi_- \quad , \quad (83)$$

we obtain,

$$\begin{aligned} E = & \int d^2x \left(\frac{3}{2} \langle D_i \phi_+ \rangle^2 + \frac{1}{2} \langle D_i \phi_- \rangle^2 + \langle \bar{D} \zeta_1 \rangle^2 + \langle D \zeta_2 \rangle^2 + \langle Z_q, Z_q \rangle \right) \\ & + \int d^2x \left(\frac{1}{2} \langle \Phi - B \rangle^2 + \gamma_r \left(\langle [\phi_1, \phi_2] \rangle^2 + \langle [\phi^1, \zeta_2] \rangle^2 + \langle [\phi^2, \zeta_1] \rangle^2 \right) \right) . \end{aligned}$$

Therefore, the BPS equations are,

$$D_i \phi_- = 0 \quad , \quad D_i \phi_+ = 0 \quad , \quad (84)$$

$$v_c \zeta_2 - [\phi^2, \zeta_2] = 0 \quad , \quad \phi^2 = 3\phi_+ + \phi_- \quad , \quad (85)$$

$$v_c \zeta_1 - [\phi^1, \zeta_1] = 0 \quad , \quad \phi^1 = 3\phi_+ - \phi_- \quad , \quad (86)$$

$$[\phi_1, \phi_2] = 0 \quad , \quad [\phi^1, \zeta_2] = 0 \quad , \quad [\phi^2, \zeta_1] = 0 \quad . \quad (87)$$

$$D \zeta_2 = 0 \quad , \quad \bar{D} \zeta_1 = 0 \quad , \quad (88)$$

$$B = g \left(v_c \phi_- - \left([\zeta_2, \zeta_2^\dagger] - [\zeta_1, \zeta_1^\dagger] \right) \right) \quad . \quad (89)$$

4 Center vortex ansatz

In order to propose a center vortex ansatz, it would be useful having a parametrization analogous to the simple $U(1)$ case, where evidencing the modulus and the phase of the complex Higgs field, $\rho e^{i\chi}$, accompanied by the gauge field, $a \partial_i \chi$, permits the implementation of boundary conditions. For this purpose, we could initially determine whether the asymptotic vacua are of the form given in eq. (33) or (34), and then look for the mapping S (the non Abelian phase) such that,

$$S(\vec{\alpha}_q \cdot \vec{H})S^{-1} \quad , \quad SE_{\alpha_q}S^{-1} \quad , \quad (90)$$

respectively

$$S(-\vec{\alpha}_q \cdot \vec{H})S^{-1} \quad , \quad S(-E_{\alpha_q}^T)S^{-1} \quad , \quad (91)$$

is the closest local basis to the field configuration ϕ_q, ζ_q . The “polar” decomposition is then,

$$\phi_q = S F_q S^{-1} \quad , \quad \zeta_q = S Z_q S^{-1} . \quad (92)$$

The notion of closest mapping can be obtained by following similar steps to those used when defining adjoint Laplacian center gauges [30]. For example, in $SU(2)$, we can take $\psi_1 = \sqrt{2} \phi_1$, together with ψ_2, ψ_3 (obtained from ζ_1 using eq. (68)), and expand these fields in the T_A basis,

$$\psi_A = \psi_{AB} T_B \quad , \quad \psi_{AB} = \langle \psi_A, T_B \rangle , \quad (93)$$

$A = 1, 2, 3$. The real elements ψ_{AB} form a 3×3 matrix Ψ , for which a polar decomposition exists,

$$\Psi = Q R , \quad (94)$$

where $R \in SO(3)$ and Q is real symmetric and positive semidefinite. The closest orthogonal matrix to Ψ is R , then the closest orthonormal basis to $\{\psi_A\}$ is given by,

$$n_A = R_{AB} T_B = S T_A S^{-1} , \quad (95)$$

where S is defined up to a global center element. That is, for $SU(2)$ adjoint Higgs fields, the “modulus and phase” decomposition is,

$$\psi_A = S(Q_{AB} T_B) S^{-1} , \quad (96)$$

which can be translated back to ϕ_1, ζ_1 -language.

With regard to the gauge field, we note that on any simply connected region, which does not contain the pointlike defects of the local basis, the Higgs field ansatz looks as a gauge transformation. Therefore, in that region, the field equations would be simplified by representing the smooth A_i as a gauge transformation of a vector field \mathcal{A}_i . However, in the defining representation, S is in general discontinuous on some curves, as it changes by a center element when we go around a center vortex. Therefore, on $R^2 - \{\text{pointlike defects}\}$, the ansatz,

$$S \mathcal{A}_i S^{-1} + \frac{i}{g} S \partial_i S^{-1} , \quad (97)$$

cannot work, as it contains a contribution (I_i) concentrated at the points where S^{-1} is discontinuous. There are three equivalent possibilities to circumvent this problem.

- proceed as in ref. [31, 32], proposing the parametrization,

$$A_i = S \mathcal{A}_i S^{-1} + \frac{i}{g} S \partial_i S^{-1} - I_i , \quad (98)$$

- proceed as in [33, 34], to write

$$A_i = (\mathcal{A}_i^A - C_i^A) n_A , \quad (99)$$

where C_i^A only depends on the local colour frame (95).

- work with the fields mapped into the adjoint representation⁴,

$$Ad(A_i) = RAd(\mathcal{A}_i)R^{-1} + \frac{i}{g}R\partial_i R^{-1} \quad , \quad Ad(\mathcal{A}_i) = \mathcal{A}_i^A M_A , \quad (100)$$

$$Ad(\psi_q) = RAd(P_q)R^{-1} \quad , \quad Ad(\zeta_\alpha) = RAd(P_\alpha)R^{-1} , \quad (101)$$

where we used,

$$R_{AB} M_B = RM_A R^{-1} . \quad (102)$$

Here, we shall use the third possibility. The advantage of the second and third options is that n_A and R contain at most pointlike defects, as they are always single-valued when we go around a loop. Then, the $R\partial_i R^{-1}$ term does not introduce delta distributions concentrated on curves, and a smooth $Ad(A_i)$ ansatz can be implemented with $Ad(\mathcal{A}_i)$ satisfying appropriate boundary conditions at the vortex guiding centers.

It is important to underline that, in the ansatz (100), $Ad(A_i)$ is *not* a gauge transformation of $Ad(\mathcal{A}_i)$. The magnetic field B is given by,

$$Ad(B) = RAd(\mathcal{B})R^{-1} + \frac{i}{g}R[\partial_1, \partial_2]R^{-1} , \quad (103)$$

$$\mathcal{B} = \partial_1 \mathcal{A}_2 - \partial_2 \mathcal{A}_1 - ig[\mathcal{A}_1, \mathcal{A}_2] , \quad (104)$$

where the last term in eq. (103) is concentrated at the vortex guiding centers. The profiles \mathcal{A}_i , F_q and Z_q , must be such that B and the Higgs fields be well-defined and smooth everywhere, and satisfy the desired asymptotic behaviour. For a single center vortex, with charge z modulo N , we can impose in the asymptotic region,

$$\mathcal{A}_i \rightarrow 0 \quad , \quad S \rightarrow e^{i\varphi \vec{\beta} \cdot \vec{T}} , \quad (105)$$

where $\vec{\beta}$ satisfies eq. (37). When minimizing the energy, the extension of $R = Ad(S)$ from the asymptotic region to the vortex core should not only contemplate keeping R along a Cartan direction but also other possibilities. In this regard, note that for

$$\vec{\beta} - \vec{\beta}_0 = 2N\vec{\gamma} \quad , \quad \vec{\gamma} \in \Lambda(Ad(SU(N))) , \quad (106)$$

⁴The matrices M_A are generators of the adjoint representation.

where $\Lambda(\dots)$ represents the lattice of weights of the adjoint representation (or root lattice), it is always possible to obtain a map $R(r, \varphi)$ verifying,

$$R(r, \varphi) = \begin{cases} e^{i\varphi \vec{\beta} \cdot \vec{M}}, & r > r_m \\ e^{i\varphi \vec{\beta}_0 \cdot \vec{M}}, & r < r_0, \end{cases} \quad (107)$$

that is smooth for $r \geq r_0$. This map can be constructed as $R = e^{i\varphi \vec{\beta}_0 \cdot \vec{M}} R_0$, with

$$R_0(r, \varphi) = \begin{cases} e^{i\varphi 2N \vec{\gamma} \cdot \vec{M}}, & r > r_m \\ I, & r < r_0. \end{cases} \quad (108)$$

Note that R_0 always exists as $e^{i\varphi 2N \vec{\gamma} \cdot \vec{T}}$ is a closed path in $SU(N)$, and therefore $e^{i\varphi 2N \vec{\gamma} \cdot \vec{M}}$ is topologically trivial in $Ad(SU(N))$. Different magnetic weights $\vec{\beta}_0$ imply different types of defect and profile function behaviours at $r \rightarrow 0$. For example, an asymptotic behaviour with $z = 0$ is described by any $\vec{\beta} \in 2N \Lambda(Ad(SU(N)))$. All these values can be extended to $\vec{\beta}_0 = 0$. For this choice, $R(r, \varphi)$ contains no defect at the origin and the minimization process will simply return a trivial result, corresponding to a pure gauge transformation of the vacuum configuration. For $z = \pm 1$, i.e. $\vec{\beta} = 2N \vec{w} + 2N \vec{\gamma}$, where \vec{w} is a weight of the defining representation or its conjugate, there is no manner to avoid a defect at $r \rightarrow 0$. The energy is expected to be minimized by $\vec{\beta}_0 = 2N \vec{w}$, as in this case some of the basis components will only give one turn when we go around a small circle centered at $r = 0$.

5 BPS center vortices

At the critical point, to solve the $SU(2)$ and $SU(3)$ BPS equations, it will be enough to consider,

$$\phi_q = v_c S(\vec{\alpha}_q \cdot \vec{H}) S^{-1}, \quad \zeta_q = u S E_{\alpha_q} S^{-1}, \quad (109)$$

(and a similar expression for the conjugate sector). The possible non Abelian phases S are such that $R = Ad(S)$ behaves as in eq. (107). As we will see, the A_i parametrization in terms of \mathcal{A}_i and S together with the BPS equations imply,

$$\mathcal{A}_i = c_i \vec{\delta} \cdot \vec{H}, \quad (110)$$

where H_q denote the Cartan generators. Then, from eqs. (98)-(100), for $r < r_0$ the gauge field is,

$$A_i = c_i \vec{\delta} \cdot \vec{H} + \frac{1}{g} \partial_i \varphi \vec{\beta}_0 \cdot \vec{H}, \quad (111)$$

and in order to obtain a regular magnetic field, we must have $\vec{\delta} = \pm \vec{\beta}_0$ and $c_i \rightarrow \mp \frac{1}{g} \partial_i \varphi$, when $r \rightarrow 0$.

5.1 $\mathfrak{su}(2)$

For nonzero ζ , eq. (64) implies,

$$\phi = \frac{v_c}{\sqrt{2}} S H_1 S^{-1} \quad , \quad \zeta = u S E_{\alpha_1} S^{-1} \quad , \quad (112)$$

where the possible magnetic weights β in eq. (107) are $\beta = q\sqrt{2}$, $q \in \mathbb{Z}$. Now, using any of the parametrizations (98)-(100), eq. (63) gives,

$$D_i(\mathcal{A})(H_1) = 0 \quad \text{or}, \quad [\mathcal{A}_i, H_1] = 0 \quad , \quad (113)$$

whose solution is,

$$\mathcal{A}_i = c_i \beta_0 H_1 \quad , \quad (114)$$

(the case $\delta = -\beta_0$ is discussed at the end). Similarly, eq. (65) becomes,

$$D(\mathcal{A})(u E_{\alpha_1}) = 0 \quad \text{or}, \quad (\partial_1 + i\partial_2)u E_{\alpha_1} - i g u [\mathcal{A}_1 + i\mathcal{A}_2, E_{\alpha_1}] = 0 \quad . \quad (115)$$

Thus, joining this information, we obtain,

$$\frac{\beta_0}{\sqrt{2}} (c_1 + i c_2) = \frac{1}{2g} (\partial_2 h - i \partial_1 h) \quad , \quad u = v_c e^{h/2} \quad . \quad (116)$$

$$Ad(\mathcal{B}) = (\partial_1 c_2 - \partial_2 c_1) \beta_0 Ad(H_1) = -\frac{1}{\sqrt{2}g} (\partial_1^2 + \partial_2^2) h Ad(H_1) \quad , \quad (117)$$

where we have changed the variables from u to $h = 2 \ln(u/v_c)$, as is usually done in the $U(1)$ case. Therefore, eqs. (66) and (103) imply,

$$((\partial_1^2 + \partial_2^2)h + g^2 v_c^2 (1 - e^h)) R Ad(H_1) R^{-1} = i\sqrt{2} R [\partial_1, \partial_2] R^{-1} \quad . \quad (118)$$

The second member is obtained from eq. (107),

$$i R [\partial_1, \partial_2] R^{-1} = \beta_0 [\partial_1, \partial_2] \varphi Ad(H_1) \quad . \quad (119)$$

As is well-known, although for $j_i = \partial_i \varphi$ the quantity $\partial_2 j_3 - \partial_3 j_2 = [\partial_2, \partial_3] \varphi$ seems to vanish, it is in fact concentrated at $x^2 = x^3 = 0$, where $e^{i\varphi}$ contains a defect. Namely,

$$\partial_1 j_2 - \partial_2 j_1 = 2\pi \delta^{(2)}(x^1, x^2) \quad . \quad (120)$$

This can be checked using Stokes' theorem. Then, we get,

$$(\partial_1^2 + \partial_2^2) h + g^2 v_c^2 (1 - e^h) = 2\pi\sqrt{2}\beta_0 \delta^{(2)}(x^1, x^2) \quad . \quad (121)$$

For q even, the asymptotic behaviour $S = e^{iq\sqrt{2}\varphi H_1}$, $R = e^{iq\sqrt{2}\varphi Ad(H_1)}$, on the circle $r \rightarrow \infty$, can be continuously changed to a behaviour characterized

by $\beta_0 = 0$, as r is varied from ∞ to 0. The absence of defects will lead to a trivial pure gauge solution for the BPS equations. On the other hand, for q odd, the asymptotic behavior can be changed to $\beta_0 = +\sqrt{2}$, as well as $\beta_0 = -\sqrt{2}$. For these values, the frame components $ST_{\alpha_1}S^{-1}$, $ST_{\bar{\alpha}_1}S^{-1}$,

$$T_\alpha = \frac{1}{\sqrt{2}}(E_\alpha + E_{-\alpha}) \quad , \quad T_{\bar{\alpha}} = \frac{1}{\sqrt{2}i}(E_\alpha - E_{-\alpha}) \quad , \quad (122)$$

rotate only once, when we go close to and around the origin. The solution to eq. (121) is well-defined for $\beta_0 = +\sqrt{2}$, while it is ill-defined for $\beta_0 = -\sqrt{2}$. In the latter case, the well-defined solution is obtained using the conjugate ansatz (see a similar discussion in 5.2),

$$\phi = \frac{v_c}{\sqrt{2}} S(-H_1)S^{-1} \quad , \quad \zeta = u S(-E_{\alpha_1}^T)S^{-1} \quad , \quad \mathcal{A}_i = c_i (-\sqrt{2})H_1 \quad . \quad (123)$$

In $SU(2)$, both the vortex and its conjugate satisfy,

$$S(\varphi + 2\pi) = -S(\varphi) \quad , \quad (124)$$

so they are equivalent objects.

5.2 $\mathfrak{su}(3)$

The equations (85)-(87) imply,

$$\phi_q = v_c S(\vec{\alpha}_q \cdot \vec{H})S^{-1} \quad , \quad \zeta_q = u_q S E_{\alpha_q} S^{-1} \quad (125)$$

($q=1,2$) so that imposing eq. (84), we obtain,

$$D_i(\mathcal{A})(\vec{\alpha}_q \cdot \vec{H}) = 0 \quad \text{or,} \quad [\mathcal{A}_i, \vec{\alpha}_q \cdot \vec{H}] = 0 \quad . \quad (126)$$

This means that \mathcal{A}_i is in the Cartan subalgebra. Taking $\vec{\delta} = +\vec{\beta}_0$,

$$\mathcal{A}_i = c_i \vec{\beta}_0 \cdot \vec{H} \quad , \quad (127)$$

eq. (88) gives,

$$(\partial_1 + i\partial_2)u_2 E_{\alpha_2} - ig(c_1 + ic_2)u_2 [\vec{\beta}_0 \cdot \vec{H}, E_{\alpha_2}] = 0 \quad , \quad (128)$$

$$(\partial_1 - i\partial_2)u_1 E_{\alpha_1} - ig(c_1 - ic_2)u_1 [\vec{\beta}_0 \cdot \vec{H}, E_{\alpha_1}] = 0 \quad . \quad (129)$$

Then, we get,

$$(\vec{\beta}_0 \cdot \vec{\alpha}_2)(c_1 + ic_2) = \frac{1}{2g}(\partial_2 h_2 - i\partial_1 h_2) \quad , \quad (130)$$

$$(\vec{\beta}_0 \cdot \vec{\alpha}_1) (c_1 - ic_2) = -\frac{1}{2g} (\partial_2 h_1 + i\partial_1 h_1) , \quad (131)$$

$$\mathcal{B} = -\frac{1}{2g} (\partial_1^2 + \partial_2^2) h \frac{\vec{\beta}_0 \cdot \vec{H}}{\vec{\beta}_0 \cdot \vec{\alpha}_2} , \quad (132)$$

where $u_i = v_c e^{h_i/2}$. In addition, eq. (89) reads,

$$\begin{aligned} iR[\partial_1, \partial_2]R^{-1} &= \\ &= R \text{Ad} \left(-g\mathcal{B} + g^2 v_c^2 (\vec{\alpha}_2 - \vec{\alpha}_1) \cdot \vec{H} - g^2 v_c^2 (\vec{\alpha}_2 e^{h_2} - \vec{\alpha}_1 e^{h_1}) \cdot \vec{H} \right) R^{-1} , \end{aligned}$$

while for a single vortex, eq. (107) implies,

$$iR[\partial_1, \partial_2]R^{-1} = [\partial_1, \partial_2]\varphi \text{Ad}(\vec{\beta}_0 \cdot \vec{H}) = 2\pi \delta^{(2)}(x^1, x^2) \text{Ad}(\vec{\beta}_0 \cdot \vec{H}) . \quad (133)$$

Putting this information together, we arrive at,

$$-g\mathcal{B} + g^2 v_c^2 (\vec{\alpha}_2 - \vec{\alpha}_1) - g^2 v_c^2 (\vec{\alpha}_2 e^{h_2} - \vec{\alpha}_1 e^{h_1}) = 2\pi \delta^{(2)}(x^1, x^2) \vec{\beta}_0 . \quad (134)$$

Let us consider the case where $\vec{\beta}_0$ is associated with a weight of the defining representation. Noting that $\mathcal{B} = (\partial_1 c_2 - \partial_2 c_1) \vec{\beta}_0 \cdot \vec{H}$ and $\vec{\alpha}_2 - \vec{\alpha}_1 = \frac{1}{2} \vec{\beta}_2$, in order to have a nontrivial solution we are led to $\vec{\beta}_0 = \pm \vec{\beta}_2$. For these cases, $\vec{\beta}_0 \cdot \vec{\alpha}_2 = -\vec{\beta}_0 \cdot \vec{\alpha}_1$, the equations (130) and (131) give $h_1 = h_2 = h$, $u_i = u = e^h$, and both sides of eq. (134) turn out to be oriented along the same direction. Under these conditions, we obtain,

$$(\vec{\beta}_0 \cdot \vec{\alpha}_2)^{-1} (\partial_1^2 + \partial_2^2) h \vec{\beta}_0 + g^2 v_c^2 (1 - e^h) \vec{\beta}_2 = 4\pi \delta^{(2)}(x^1, x^2) \vec{\beta}_0 . \quad (135)$$

That is, for $\vec{\beta}_0 = +\vec{\beta}_2$ ($\vec{\beta}_0 \cdot \vec{\alpha}_2 = +1$),

$$(\partial_1^2 + \partial_2^2) h + g^2 v_c^2 (1 - e^h) = 4\pi \delta^{(2)}(x^1, x^2) . \quad (136)$$

On the other hand, the choice $\vec{\beta}_0 = -\vec{\beta}_2$ would imply,

$$(\partial_1^2 + \partial_2^2) h + g^2 v_c^2 (1 - e^h) = -4\pi \delta^{(2)}(x^1, x^2) . \quad (137)$$

The second choice does not lead to well-defined Higgs fields. In effect, while close to the origin eq. (136) gives $h \sim 2 \ln r$, $u = e^{h/2} \sim r$, producing single-valued Higgs fields (and $c_i \sim -\frac{1}{g} \partial_i \varphi$), eq. (137) gives $h \sim -2 \ln r$, $u = e^{h/2} \sim 1/r$. However, it is easy to see that the new ansatz obtained from (125) by the replacement,

$$\vec{\alpha}_q \cdot \vec{H} \rightarrow -\vec{\alpha}_q \cdot \vec{H} \quad , \quad E_{\alpha_q} \rightarrow -E_{\alpha_q}^T , \quad (138)$$

solves the BPS equations with a well-defined h satisfying eq. (136), provided we choose $\vec{\beta}_0 = -\vec{\beta}_2$. Other weights can be obtained by replacing in eq. (125) (resp. eq. (138)),

$$\vec{\alpha}_q \rightarrow \vec{\alpha}_q^W, \quad (139)$$

where W is a Weyl transformation. The solutions will be characterized by the gauge field behaviour (107), with $\vec{\beta}_0 \rightarrow \vec{\beta}_0^W = \vec{\beta}_2^W$ (resp. $\vec{\beta}_0 \rightarrow \vec{\beta}_0^W = -\vec{\beta}_2^W$) (and $\vec{\beta} \rightarrow \vec{\beta}^W$). Then, these solutions are characterized by the weights of the defining representation, $\vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3$, and their conjugates, $-\vec{\beta}_1, -\vec{\beta}_2, -\vec{\beta}_3$. As the mappings S satisfy,

$$S(\varphi + 2\pi) = e^{\pm i2\pi/3} S(\varphi), \quad (140)$$

they correspond to center vortices with the minimum charges $z = \pm 1$.

6 Conclusions

In this article we presented Yang-Mills-Higgs nonrelativistic models with $SU(N) \rightarrow Z(N)$ SSB pattern that accept BPS center vortex equations (for $N = 2, 3$).

For this purpose, we initially proposed a class of $SU(N)$ Lorentz invariant models containing real and complex adjoint Higgs fields, that can be labelled by the simple roots of the $\mathfrak{su}(N)$ Lie algebra. The Higgs potential is such that its minimization returns a set of conditions that essentially define a Chevalley basis. The space of vacua also contains a trivial symmetry preserving point, where the Higgs fields vanish, separated from the SSB points by a potential barrier.

Next, we introduced a nonrelativistic interaction term so as to obtain a set of BPS equations. This is a term that tends to align, in the Lie algebra, the magnetic field and one of the Higgs fields. Finally, we obtained some solutions. For example, the $Z(3)$ vortices come in three colours (the weights of the defining representation), which are physically equivalent, and three anticolours, obtained from an ansatz based on the conjugate representation.

Generally, BPS equations are derived by working on the energy functional, which is a local object, and obtaining a bound that only depends on some topological charge. For $U(1)$ vortices, the bound is given by the magnetic flux. This is a topological term that can be written locally, by means of a flux density. On the other hand, for center vortices, the flux concept is given by the asymptotic behaviour of the gauge invariant Wilson loop, a nonlocal object that may not arise in the calculation. For this reason, the search for BPS equations led us to consider the alignment interaction.

After completing the squares, the energy is always greater than or equal to zero. Thus, BPS center vortices are characterized by an exact compensation between the positive definite part of the energy functional (kinetic energy plus Higgs potential) and the contribution originated from alignment.

Similarly to the minima of the Higgs potential, the BPS equations have trivial solutions with vanishing Higgs fields (and pure gauge fields) and a sector where the asymptotic fields are in SSB vacua. Although the BPS solutions have vanishing energy, no finite energy configurations continuously interpolating the center vortex ($z = \pm 1$) and the trivial configuration ($z = 0$) exist. In other words, there is an energy barrier for the continuous deformation of one configuration into the other. The general solution to the BPS equations was written in terms of a reduced set of profile functions and a mapping $R(S)$ in the adjoint representation of $SU(N)$. The mapping $S \in SU(N)$, contains information about the asymptotic Wilson loop and the set of possible defects at the vortex guiding centers, which determine the behaviour of the profile functions.

In spite of the Abelian looking profile functions obtained, we would like to underline two important differences. As the number of BPS center vortices is increased, the energy continues to vanish. This is in contrast to the $U(1)$ case, where the energy increases linearly with the number of vortices, a property that is modified below and above the critical coupling, implying either attractive or repulsive forces. In addition, the topological properties of the adjoint representation of $SU(N)$ modify the relation between asymptotic phases and defects. A $U(1)$ asymptotic phase implies a unique type of pointlike defect, and a unique order for the zero of the corresponding Higgs profile function. On the other hand, for an asymptotic non Abelian phase, many extensions to reach a pointlike defect are possible, with corresponding conditions on the profile functions. For $Z(N)$ -charge equal to ± 1 , a defect is always present, while for vanishing $Z(N)$ -charge the defects in $R(S)$ can be avoided, and the lowest energy solution simply corresponds to a regular gauge transformation of the SSB vacua. When leaving the critical point, the new energetics, topology, and field content are expected to modify the forces between center vortices, as compared with the $U(1)$ case. This may be a possibility worth exploring.

Summarizing, the search for BPS bounds is among the preferred analytical tools to understand topological objects. In this manuscript, we showed what would be the situation in the context of center vortex $(2+1)$ d models: they become nonrelativistic. Then, although BPS center vortices are not directly physically relevant, they could provide a useful concept when embarking on numerical simulations. The presence of a BPS point in the extended parameter space could serve as a check of the numerical analysis

when moving away from the physically relevant non BPS Lorentz invariant confining models.

Acknowledgements

The Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq-Brazil), the Fundação de Amparo a Pesquisa do Estado do Rio de Janeiro (FAPERJ), and the Proppi-UFF are acknowledged for their financial support.

Appendix A: Cartan decomposition of \mathfrak{g}

A compact connected simple Lie algebra \mathfrak{g} can be decomposed in terms of hermitian Cartan generators H_q , $q = 1, \dots, r$, which generate a Cartan subgroup H , and off-diagonal generators E_α , or root vectors. The latter are labelled by a system of roots $\vec{\alpha} = (\alpha_1, \dots, \alpha_r)$. They satisfy,

$$[H_q, H_p] = 0 \quad , \quad [H_q, E_\alpha] = \alpha_q E_\alpha \quad , \quad [E_\alpha, E_\alpha^\dagger] = \langle E_\alpha, E_\alpha \rangle H_\alpha \quad , \quad (141)$$

where, for every root $\vec{\alpha}$, H_α is defined by,

$$\langle H_\alpha, H_q \rangle = \vec{\alpha}|_q \quad . \quad (142)$$

The rank of $\mathfrak{su}(N)$ is $r = N - 1$, and its dimension is $d = N^2 - 1$. The weights of the defining representation, can be ordered according to,

$$\vec{w}_1 > \vec{w}_2 > \dots > \vec{w}_N \quad , \quad (143)$$

so that the positive and simple roots are, respectively,

$$\vec{\alpha}_{qp} = \vec{w}_q - \vec{w}_p \quad , \quad q < p \quad , \quad \vec{\alpha}_q = \vec{w}_q - \vec{w}_{q+1} \quad . \quad (144)$$

Finally, recalling that the fundamental weights $\vec{\Lambda}^q$ are defined by,

$$\frac{2 \vec{\alpha}_q \cdot \vec{\Lambda}^p}{\vec{\alpha}_{(q)} \cdot \vec{\alpha}_{(q)}} = \delta_q^p \quad , \quad (145)$$

the $\vec{\lambda}^q$ basis in eq. (18) can be written as,

$$\vec{\lambda}^q = 2N \vec{\Lambda}^q \quad , \quad (146)$$

with,

$$\vec{\Lambda}^1 = \vec{w}_1 \quad , \quad \vec{\Lambda}^2 = \vec{w}_1 + \vec{w}_2 \quad , \quad \vec{\Lambda}^3 = \vec{w}_1 + \vec{w}_2 + \vec{w}_3 \quad , \quad \dots \quad (147)$$

References

- [1] W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. **42** (1979) 1698.
- [2] E. Fradkin, Field Theories of Condensed Matter Physics (2nd edition, Cambridge University Press, 2013).
- [3] T. H. R. Skyrme, Proc. R. Soc. Lond. **260** (1961) 127.
- [4] C. J. Houghton, N. S. Manton, and P. M. Sutcliffe, Nuc. Phys. **B510** (1998) 507.
- [5] N. Manton and P. Sutcliffe, Topological Solitons (Cambridge University Press, 2004).
- [6] M. Baker, J. S. Ball and F. Zachariasen, Phys. Rev. **D51** (1995) 1968.
- [7] M. Baker, J. S. Ball, N. Brambilla, G. M. Prosperi and F. Zachariasen, Phys. Rev. **D54** (1996) 2829, erratum, *ibid.* **D56** (1997) 2475.
- [8] S. Maedan and T. Suzuki, Progr. Theor. Phys. **81** (1989) 229.
- [9] H. Shiba, S. Kamisawa, Y. Matsubara and T. Suzuki, Nucl. Phys. **B389** (1993) 563.
- [10] M. N. Chernodub and D. A. Komarov, JETP Lett. **68** (1998) 117.
- [11] K. Konishi and L. Spanu, Int. J. Mod. Phys. **A18** (2003) 249.
- [12] H. J. de Vega, Phys. Rev. **D18** (1978) 2932.
- [13] H.J. de Vega and F.A. Schaposnik, Phys. Rev. Lett. **56** (1986) 2564.
- [14] H. J. de Vega and F. A. Schaposnik, Phys. Rev. **D34** (1986) 3206.
- [15] J. Heo and T. Vachaspati, Phys. Rev. **D58** (1998) 065011.
- [16] L. E. Oxman, J. High Energy Phys. **03** (2013) 038.
- [17] B. Ketzer, PoS (QNP2012) 025.
- [18] J. J. Dudek, R. G. Edwards, M. J. Peardon, D. G. Richards and C. E. Thomas, Phys. Rev. Lett. **103** (2009) 262001.
- [19] J. J. Dudek and R. G. Edwards, arXiv:1201.2349.

- [20] P. Goddard, J. Nyuts and D. Olive, Nucl. Phys. **B125** (1977) 1.
- [21] A. Hanany and D. Tong, JHEP **0307** (2003) 037.
- [22] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B 673 (2003) 187.
- [23] D. Tong, Phys. Rev. **D69** (2004) 065003.
- [24] D. Tong, Ann. of Phys. **324** (2009) 30.
- [25] K. Konishi, Lect. Notes Phys. **737** (2008) 471.
- [26] S.B. Bradlow, O. Garcia-Prada, Non-abelian monopoles and vortices, arXiv:alg-geom/9602010.
- [27] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory* (Springer, New York, 1972).
- [28] B. C. Hall, *Lie groups, Lie Algebras and Representations* (Springer, New York, 2003).
- [29] H. Georgi, *Lie Algebras in Particle Physics*, Frontiers in Physics.
- [30] Ph. de Forcrand and M. Pepe, Nucl. Phys. **B598** (2001) 557.
- [31] M. Engelhardt, H. Reinhardt, Nucl. Phys. **B567** (2000) 249.
- [32] H. Reinhardt, Nucl. Phys. **B628** (2002) 133.
- [33] L. E. Oxman, JHEP **12** (2008) 089.
- [34] L. E. Oxman JHEP **07** (2011) 078.